Efficient evaluation of the Brauer–Manin obstruction

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Abstract

The Brauer–Manin obstruction is a concept which has been very effective in finding counter-examples to the Hasse principle, that is, sets of polynomial equations which have solutions in every completion of the rational numbers but have no rational solutions. The standard way of calculating the Brauer–Manin obstruction involves listing all the p-adic solutions to some accuracy, at finitely many primes p; this is a process which may be time-consuming. The result described in this article shows that, at some primes, we do not need to list all p-adic solutions, but only those lying over a closed subset; and, at other primes, we need only to list solutions modulo p.

1. *Introduction*

The Brauer–Manin obstruction was introduced by Manin [12, 13] to explain why some systems of algebraic equations over the rational numbers $\mathbb{Q}$, while having solutions everywhere locally, fail to have global solutions. The obstruction is based on the Brauer group of the associated variety [see 8]. If we denote by $V(\mathbb{A}_\mathbb{Q})$ the set of adelic points of the variety V, then we may define

$$V(\mathbb{A}_\mathbb{Q})^B := \Big\{ (x_v) \in V(\mathbb{A}_\mathbb{Q}) \quad \Big| \quad \sum_v \operatorname{inv}_v \mathcal{A}(x_v) = 0 \text{ for all } \mathcal{A} \text{ in } B \Big\}$$

to be the set of adelic points orthogonal to a subset B of the Brauer group $\operatorname{Br} V$, the sum here being taken over all places v of $\mathbb{Q}$. Manin's observation was that, by a standard result in class field theory, any rational points must be contained in $V(\mathbb{A}_\mathbb{Q})^{\operatorname{Br} V}$, through the usual diagonal inclusion of $\mathbb{Q}$ in $\mathbb{A}_\mathbb{Q}$. The point is that $V(\mathbb{A}_\mathbb{Q})^{\operatorname{Br} V}$ is sometimes amenable to computation, giving a potential way to prove the non-existence of rational points on V.

One class of varieties on which one can often compute the Brauer–Manin obstruction is that of surfaces with finitely generated Picard group. This article is primarily aimed at computing the obstruction on such surfaces, though the results are true in a wider context. This class includes rational varieties such as cubic surfaces [16], pencils of Severi–Brauer varieties [3], intersections of two quadrics [4, 5] and some pencils of curves of genus 1 [6]. For some of these surfaces, it is known that the Brauer–Manin obstruction is the *only* obstruction to the existence of rational points: in those cases, computing the obstruction gives a definite answer to the question of whether there exist any rational points on the surface. The Brauer–Manin obstruction has also recently been used to obtain interesting results on curves [7].

In this article we address the problem of evaluating the obstruction associated to a given Azumaya algebra $\mathcal{A}$. That is, we wish to compute the set $V(\mathbb{A}_{\mathbb{Q}})^{\mathcal{A}}$. The usual method for this is as follows. We consider the local invariant function

$$V(\mathbb{Q}_v) \to \mathbb{Q}/\mathbb{Z} \quad : \quad x \mapsto \mathrm{inv}_v\, \mathcal{A}(x)$$

separately for each v. For all but finitely many v, depending on V and $\mathcal{A}$, this function is zero. At each remaining place, the function is continuous, hence locally constant. By listing all the points of $V(\mathbb{Q}_v)$ to sufficient accuracy, we can evaluate the local invariant everywhere. We then combine this information to obtain $V(\mathbb{A}_{\mathbb{Q}})^{\mathcal{A}}$.

However, listing all points on $V(\mathbb{Q}_v)$ to sufficient accuracy to compute the local invariant at each is a time-consuming process. The result proved in this article drastically reduces the amount of work needed to compute the Brauer–Manin obstruction. We assume that the Azumaya algebra is split by a finite extension of the base field – that is, it belongs to the so-called *algebraic* Brauer group $\mathrm{Br}_1(V)$. The reduction of V modulo p falls into one or more irreducible components, which we assume to be reduced.

Suppose that $\mathcal{A}$ is split by an unramified extension of $\mathbb{Q}_v$; then we prove that, at most points P of $V(\mathbb{Q}_p)$ lying above only one component, the invariant depends only on which component that is. This means that the listing of points is now only needed on the intersections of distinct components, and possibly on the non-regular subschemes of components. In this case, the arguments leading to this result are very similar to those described by Harari [10, Section 2.4].

At primes where the splitting extension is tamely ramified, we show that this approach also gives a very useful result: at most of the points, it is enough to know the point modulo p in order to calculate the invariant there.

THEOREM 1. *Let X be a smooth complete variety over $\mathbb{Q}_p$ and $\mathcal{A}$ an Azumaya algebra on X split by a finite Galois extension $K/\mathbb{Q}_p$. Fix a proper model $\mathcal{X}/\mathbb{Z}_p$ of X and let*

$$q : X(\mathbb{Q}_p) \to \mathcal{X}(\mathbb{F}_p)$$

be the reduction map. Let Y be an irreducible component of multiplicity 1 of the special fibre $\tilde{X}$. Then there exists a dense open subset U of Y such that

- *if $K/\mathbb{Q}_p$ is unramified, then* $\mathrm{inv}_p\, \mathcal{A}$ *is constant on $q^{-1}U(\mathbb{F}_p)$;*
- *if $K/\mathbb{Q}_p$ is tamely ramified, then* $\mathrm{inv}_p\, \mathcal{A}$ *is constant on $q^{-1}\{x\}$, for each $x \in U(\mathbb{F}_p)$.*

We will define the subscheme U as follows:

- if $K/\mathbb{Q}_p$ is unramified, U consists of those points on Y which are regular as points of $\mathcal{X}$;
- if $K/\mathbb{Q}_p$ is ramified, U consists of the smooth points of Y.

It may be possible, by looking more closely at the geometry of X, to take a larger U than that defined here.

2. *Proof of Theorem 1*

If X is a variety over $\mathbb{Q}_p$ and $K/\mathbb{Q}_p$ a finite field extension, then X_K denotes the base extension of X to K. Similarly, if $\mathcal{X}$ is a scheme over $\mathbb{Z}_p$, then $\mathcal{X}_K$ denotes the base extension of $\mathcal{X}$ to $\mathcal{O}_K$.

If $\mathcal{X}$ is any scheme, then $\operatorname{Div}\mathcal{X}$ denotes the group of Cartier divisors on $\mathcal{X}$. Let Y be a reduced closed subscheme of $\mathcal{X}$; then $\operatorname{Div}_Y\mathcal{X}$ denotes the subgroup of Cartier divisors whose support is contained in Y. There is an exact sequence

$$0 \to \operatorname{Div}_Y\mathcal{X} \to \operatorname{Div}\mathcal{X} \to \operatorname{Div}(\mathcal{X}\setminus Y)$$

in which the rightmost map is in general not surjective. In particular, we will apply this when Y is the special fibre of $\mathcal{X}$; in that case, the notation $\operatorname{Div}'\mathcal{X}$ will be used for the subgroup of Cartier divisors supported on the special fibre. The exact sequence becomes

$$0 \to \operatorname{Div}'\mathcal{X} \to \operatorname{Div}\mathcal{X} \to \operatorname{Div}X.$$

We now state a hypothesis which will be satisfied by all the schemes we will consider.

Condition (*). $\mathcal{X}$ is an integral Noetherian separated scheme over $\mathbb{Z}_p$ such that
- (i) the generic fibre X is a smooth variety;
- (ii) the special fibre $\tilde{X}$ is integral;
- (iii) the sequence

$$H^2(K/\mathbb{Q}_p,\operatorname{Div}'\mathcal{X}_K) \to H^2(K/\mathbb{Q}_p,\operatorname{Div}\mathcal{X}_K) \to H^2(K/\mathbb{Q}_p,\operatorname{Div}X_K) \tag{2·1}$$

is exact.

In particular, the sequence (2·1) is exact if $\mathcal{X}_K$ is regular, for then $\operatorname{Div}\mathcal{X}_K$ is equal to the Weil divisor group and the sequence splits as a direct sum.

The scheme $\mathcal{X}$ which we will be studying to prove Theorem 1 is constructed as follows: take the given proper model for X over $\mathbb{Z}_p$, and then remove from the special fibre everything outside the open set U. This gives a scheme over $\mathbb{Z}_p$ which is regular, and so satisfies Condition (*). For, if $K/\mathbb{Q}_p$ is unramified, then the morphism $\mathcal{X}_K \to \mathcal{X}$ is étale and therefore the inverse image on $\mathcal{X}_K$ of a regular point on $\mathcal{X}$ is again regular [see 9, I, 9.2]. On the other hand, suppose that $K/\mathbb{Q}_p$ is ramified but instead that $U = \tilde{X}$ is smooth over $\mathbb{F}_p$; then $\mathcal{X}$ is smooth over $\mathbb{Z}_p$ and so $\mathcal{X}_K$ is smooth over $\mathcal{O}_K$ [9, II, 1.3]. As $\mathcal{O}_K$ is regular, it follows that $\mathcal{X}_K$ is regular [9, II, 3.1].

If $\mathcal{X}$ is a scheme which is Noetherian, separated, integral and regular in codimension 1, then $\operatorname{Div}\mathcal{X}$ is a subgroup of the group $Z^1\mathcal{X}$ of Weil divisors [see 11, II, 6.11.2]. This will be true in particular for schemes satisfying Condition (*). For all codimension-one subschemes of the generic fibre certainly have regular local rings; the only other codimension-one subscheme is the special fibre, which has a local ring generated by p, so is also regular.

The proof of Theorem 1 consists of the following steps.
- (i) We will look at the map

$$H^2(K/\mathbb{Q}_p, k(\mathcal{X}_K)^\times) \to H^2(K/\mathbb{Q}_p,\operatorname{Div}\mathcal{X}_K) \tag{2·2}$$

and show that an algebra whose image is 0 under this map, when evaluated at a point, gives rise to a 2-cocycle with values in $\mathcal{O}_K^\times$.
- (ii) We will show that every Azumaya algebra on X is equivalent to a constant algebra plus an algebra giving 0 under the map (2·2).

It is only a short argument to proceed from these results to the theorem.

2·1. *The divisor map on Azumaya algebras*

Let X be a smooth, irreducible variety over $\mathbb{Q}_p$. Let $\mathcal{R}_X$ and $\mathcal{D}_X$ denote the sheaves of rational functions and Cartier divisors respectively. Any Azumaya algebra on X can be represented by an element of $H^2(X, \mathcal{R}_X)$ which maps to 0 in $H^2(X, \mathcal{D}_X)$. If $\mathcal{A}$ is an Azumaya algebra split by a finite field extension $K/\mathbb{Q}_p$, then a stronger statement is true: $\mathcal{A}$ is represented by an element of $H^2(K/\mathbb{Q}_p, k(X_K)^\times)$ which maps to 0 in $H^2(K/\mathbb{Q}_p, \operatorname{Div} X_K)$. This follows from the Hochschild–Serre spectral sequence and the fact that $H^1(X, \mathcal{D}_X) = 0$.

Now let $\mathcal{X}$ be our model of X over $\mathbb{Z}_p$ satisfying Condition (*). Let $\operatorname{Br}(X_K/X)$ denote the subgroup of $\operatorname{Br} X$ consisting of algebras which split over the extension $K/\mathbb{Q}_p$. Since the sequence (2·1) is exact, the map (2·2) gives rise to a map

$$\phi_K : \operatorname{Br}(X_K/X) \to H^2(K/\mathbb{Q}_p, \operatorname{Div}' \mathcal{X}_K).$$

When $\mathcal{X}$ is proper and regular and $K/\mathbb{Q}_p$ is unramified, the purity theorem for the Brauer group (discussed by Grothendieck [8, III,6] and, we understand, proved by Gabber) states that an Azumaya algebra $\mathcal{A}$ on X, split by K, extends to an Azumaya algebra on $\mathcal{X}$ if and only if $\phi_K(\mathcal{A}) = 0$. In this case, evaluating $\mathcal{A}$ at a $\mathbb{Q}_p$-point of X always gives a trivial algebra: for the evaluation map factors through $\operatorname{Br} \mathbb{Z}_p = 0$.

We will relax these hypotheses and see what happens even when $K/\mathbb{Q}_p$ is ramified.

Lemma 2. *Let $V \to U$ be a finite surjective morphism of integral schemes, acted on by a group G such that $U = V/G$ and $G = \operatorname{Gal}(k(V)/k(U))$. Let α be an element of $H^2(G, k(V)^\times)$ which maps to 0 in $H^2(G, \operatorname{Div} V)$. Suppose that $\operatorname{Pic} V = 0$. Then there exists a cocycle cohomologous to α taking values in $k[V]^\times$.*

Proof. As $\operatorname{Pic} V = 0$, the sequence of G-modules

$$0 \to k[V]^\times \to k(V)^\times \to \operatorname{Div} V \to 0$$

is exact. Part of the long exact sequence in cohomology is

$$H^2(G, k[V]^\times) \to H^2(G, k(V)^\times) \to H^2(G, \operatorname{Div} V)$$

which gives the result. □

Proposition 3. *Let $\mathcal{X}$ satisfy Condition (*). Let P be a point of $X(\mathbb{Q}_p)$ extending to a $\mathbb{Z}_p$-valued point of $\mathcal{X}$. Let $\mathcal{A}$ be an Azumaya algebra on X split by a finite Galois extension $K/\mathbb{Q}_p$, and suppose that $\phi_K(\mathcal{A}) = 0$. Then $\mathcal{A}(P)$ is represented by a cocycle in $Z^2(K/\mathbb{Q}_p, \mathcal{O}_K^\times)$.*

Proof. Let Q be the unique point of X_K lying above P; let $\tilde{P}$ and $\tilde{Q}$ be the closed points associated to P and Q respectively. Consider the commutative diagram

$$\begin{array}{ccc} \mathcal{O}^\times_{\mathcal{X}_K, \tilde{Q}} & \to & \mathcal{O}_K^\times \\ \cap & & \cap \\ \mathcal{O}^\times_{\mathcal{X}_K, Q} & \xrightarrow{Q} & K^\times \\ \cap & & \\ k(\mathcal{X}_K)^\times & & \end{array}$$

where the lower horizontal map is evaluation at Q, and the upper horizontal map is evaluation at the corresponding $\mathcal{O}_K$-point. We will apply Lemma 2 to the covering $\operatorname{Spec} \mathcal{O}_{\mathcal{X}_K, \tilde{Q}} \to \operatorname{Spec} \mathcal{O}_{\mathcal{X}, \tilde{P}}$. The hypotheses are satisfied because

- $\mathcal{A}$ is Azumaya on X, so maps to 0 in $H^2(K/\mathbb{Q}_p, \operatorname{Div} X_K)$;

- $\phi_K(\mathcal{A}) = 0$; combined with the previous fact, this shows that $\mathcal{A}$ maps to 0 in $H^2(K/\mathbb{Q}_p, \operatorname{Div} \mathcal{X}_K)$, using the exactness of the sequence (2·1);
- Spec $\mathcal{O}_{\mathcal{X}_K,\tilde{Q}}$ is local and so has trivial Picard group.

The conclusion is that we can represent $\mathcal{A}$ by a cocycle taking values in $\mathcal{O}^\times_{\mathcal{X}_K,\tilde{Q}}$; the diagram above shows that evaluating this cocycle at Q gives a cocycle representing $\mathcal{A}(P)$ which takes values in $\mathcal{O}_K^\times$. □

Corollary 4. *If $K/\mathbb{Q}_p$ is unramified, then* $\operatorname{inv}_p \mathcal{A}(P) = 0$.

Proof. In this case, $H^2(K/\mathbb{Q}_p, \mathcal{O}_K^\times) = 0$. See Serre [15, V,2]. □

Corollary 5. *If $K/\mathbb{Q}_p$ is tamely ramified, then* $\operatorname{inv}_p \mathcal{A}(P)$ *depends only the reduction of P modulo p.*

Proof. In this case $H^2(K/\mathbb{Q}_p, U_K^1) = 0$, where U_K^1 is the group of units in $\mathcal{O}_K$ congruent to 1 modulo the prime ideal $\mathfrak{p}_K$. This fact follows from the observation that U_K^1 is a pro-p-group and that $[K : \mathbb{Q}_p]$ is coprime to p. So the reduction map gives an isomorphism

$$H^2(K/\mathbb{Q}_p, \mathcal{O}_K^\times) \cong H^2(K/\mathbb{Q}_p, (\mathcal{O}_K/\mathfrak{p}_K)^\times)$$

through which evaluation at P factors. □

2·2. *The divisor map on constant algebras*

In this section we will show that, under hypotheses slightly stronger than Condition (*), any Azumaya algebra on X split by K can be made to satisfy the hypotheses of Proposition 3 by adding a constant algebra, that is, an element of $\operatorname{Br}(K/\mathbb{Q}_p)$.

Proposition 6. *Let $\mathcal{X}$ satisfy Condition (*). Let $\mathcal{A}$ be an Azumaya algebra on the generic fibre X which splits in a finite Galois extension K of $\mathbb{Q}_p$. Suppose that $\mathcal{X}$ has a $\mathbb{Z}_p$-point, and that either*

- *the special fibre of $\mathcal{X}_K$ is irreducible; or*
- *$\mathcal{X}_K$ is regular; or*
- *$K/\mathbb{Q}_p$ is cyclic.*

Then there exists a constant algebra B in $\operatorname{Br}(K/\mathbb{Q}_p)$ *such that* $\phi_K(\mathcal{A} - B) = 0$.

Proof. Let P be the closed point associated to the $\mathbb{Z}_p$-point on $\mathcal{X}$. There is a unique closed point Q of $\mathcal{X}_K$ lying over P. Write $\mathcal{S}$ for Spec $\mathcal{O}_{\mathcal{X}_K,Q}$.

The short exact sequence

$$0 \to \mathcal{O}^\times_{\mathcal{X}_K,Q} \to k(\mathcal{X}_K)^\times \to \operatorname{Div} \mathcal{S} \to 0$$

gives rise to an exact sequence

$$H^2(K/\mathbb{Q}_p, k(\mathcal{X}_K)^\times) \to H^2(K/\mathbb{Q}_p, \operatorname{Div} \mathcal{S}) \to H^3(K/\mathbb{Q}_p, \mathcal{O}^\times_{\mathcal{X}_K,Q})$$

of cohomology groups. $\operatorname{Br}(X_K/X)$ is contained in this left-hand group, and its image in the middle group factors through $H^2(K/\mathbb{Q}_p, \operatorname{Div}' \mathcal{X}_K)$. It follows that the bottom row of the following commutative diagram is a complex.

$$\begin{array}{ccccc} H^2(K/\mathbb{Q}_p, K^\times) & \xrightarrow{v_K} & H^2(K/\mathbb{Q}_p, \mathbb{Z}) & \longrightarrow & H^3(K/\mathbb{Q}_p, \mathcal{O}_K^\times) \\ \downarrow & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} \\ Br(X_K/X) & \xrightarrow[\gamma]{} & H^2(K/\mathbb{Q}_p, \mathrm{Div}' \mathcal{X}_K) & \longrightarrow & H^3(K/\mathbb{Q}_p, \mathcal{O}_{\mathcal{X}_K,Q}^\times) \end{array} \tag{2·3}$$

Here α is the map $1 \mapsto (\mathfrak{p}_K)$, $\mathfrak{p}_K$ being the prime in K. The top row is easily seen to be exact.

To prove the proposition, we must show that the image of αv_K contains that of γ. In fact it is enough to show that the image of α contains that of γ: for the evaluation map at the $\mathbb{Z}_p$-point gives a left inverse to β and so β is injective. Some diagram-chasing shows that any element of $H^2(K/\mathbb{Q}_p, \mathbb{Z})$ whose image under α lies in the image of γ must lift to $H^2(K/\mathbb{Q}_p, K^\times)$. The proof will be completed by Lemma 7. □

Remark. When $K/\mathbb{Q}_p$ is cyclic and totally ramified, the proposition is trivial: the maps v_K and γ are zero.

In the statement and proof of the following lemma, the special fibre of $\mathcal{X}$ will no longer be required to be irreducible. Let Y be a component of the special fibre; then by Y_L we mean the fibre product $Y \times_{\mathcal{X}} \mathcal{X}_L$.

LEMMA 7. *Let L/K be a finite Galois extension of local fields. Let $\mathcal{X}$ be an integral scheme over $\mathcal{O}_K$ and Y an integral component of the special fibre of $\mathcal{X}$. Suppose either that Y_L is irreducible, that $\mathcal{X}$ is regular, or that L/K is cyclic. Then the map*

$$H^2(L/K, \mathbb{Z}) \to H^2(L/K, \mathrm{Div}_Y \mathcal{X}_L) \tag{2·4}$$

given by $1 \mapsto (\mathfrak{p}_L)$ is surjective.

Proof. To save on notation, we will write $\mathrm{Div}_Y \mathcal{X}_L$ for the group of Cartier divisors on $\mathcal{X}_L$ which are supported on the set-theoretic inverse image of Y.

Firstly, the lemma is true when Y_L is irreducible. In that case, we claim that the map

$$\mathbb{Z} \to \mathrm{Div}_Y \mathcal{X}_L : \quad 1 \mapsto (\mathfrak{p}_L)$$

is an isomorphism of Galois modules. It is easy to see that $\mathrm{Div}_Y \mathcal{X}_L$ is free on one generator Y', so that $(\mathfrak{p}_L) = nY'$ for some integer n. The Galois action on both groups is trivial. We must show that $n = 1$, and it is enough to do this separately for L/K unramified and for L/K totally ramified. We will write π for the morphism $\mathcal{X}_L \to \mathcal{X}$.

When L/K is unramified,

$$(\mathfrak{p}_L) = (\pi^* \mathfrak{p}_K) = \pi^* Y;$$

but, since π is étale and Y reduced, $\pi^* Y$ is reduced [9, I, 9.2] and so $n = 1$.

When L/K is totally ramified of degree e, we have

$$e(\mathfrak{p}_L) = (\pi^* \mathfrak{p}_K) = \pi^* Y$$

and so $\pi^* Y = enY'$. But π is a flat, finite map; thus $en = [L : K]$ and so $n = 1$. This completes the proof when Y_L is irreducible.

We will now show that the lemma is true when the group $\mathrm{Div}_Y \mathcal{X}_L$ of Cartier divisors is replaced by the group $Z^1_Y \mathcal{X}_L$ of Weil divisors supported on Y. In particular, this proves the lemma when $\mathcal{X}_L$ is regular: for then these two groups coincide.

Write G for $\mathrm{Gal}(L/K)$. Let Y' be an irreducible component of Y_L, and $H \subseteq G$ the decomposition group of the generic point of Y'. Then the G-module $Z^1_Y \mathcal{X}_L$ is induced from the H-module $Z^1_{Y'} \mathcal{X}_L$. For any $i \geq 0$, we have the following commutative diagram.

$$\begin{tikzcd} & H^i(H,\mathbb{Z}) \arrow[r, "\sim"] & H^i(H, Z^1_{Y'}\mathcal{X}_L) \\ H^i(G,\mathbb{Z}) \arrow[ur, "\mathrm{res}"] \arrow[r, "\delta"] & H^i(G, \mathrm{Ind}_H^G \mathbb{Z}) \arrow[u, "\sim"] \arrow[r, "\sim"] & H^i(G, Z^1_Y \mathcal{X}_L) \arrow[u, "\sim"'] \end{tikzcd} \qquad (2\cdot5)$$

Here the two vertical arrows are isomorphisms by Shapiro's Lemma; and the top horizontal arrow is an isomorphism by the argument above. The arrow labelled δ comes from the map $\mathbb{Z} \to \mathrm{Ind}_H^G \mathbb{Z}$ where, thinking of $\mathrm{Ind}_H^G \mathbb{Z}$ as $\mathrm{Hom}_H(\mathbb{Z}[G], \mathbb{Z})$, the image of 1 is the function which is identically 1. The composite map $\mathbb{Z} \to Z^1_Y \mathcal{X}_L$ is indeed the map (2·4), and moreover the composite map $H^i(G,\mathbb{Z}) \to H^i(H,\mathbb{Z})$ is none other than the restriction: see Milne [14, II, 1.27].

Taking $i = 2$ in the diagram (2·5), the restriction map is surjective: for it is the map

$$\mathrm{Hom}\,(G, \mathbb{Q}/\mathbb{Z}) \to \mathrm{Hom}\,(H, \mathbb{Q}/\mathbb{Z})$$

and $\mathbb{Q}/\mathbb{Z}$ is injective. It follows that the map (2·4) is surjective, completing this step of the proof.

Finally, suppose that G is cyclic. Then, for any G-module M, there is a surjective map

$$H^0(G, M) \to H^2(G, M)$$

given by cup-product with a chosen generator of $H^2(G, \mathbb{Z})$. There is a commutative diagram as follows.

$$\begin{tikzcd} H^0(G,\mathbb{Z}) \arrow[rr, "\sim"] \arrow[dr, "\sim"] \arrow[dd, two heads] & & H^0(G, Z^1_Y \mathcal{X}_L) \arrow[dd, two heads] \\ & H^0(G, \mathrm{Div}_Y \mathcal{X}_L) \arrow[ur, "\sim"] \arrow[dd, two heads] & \\ H^2(G,\mathbb{Z}) \arrow[rr, two heads] \arrow[dr] & & H^2(G, Z^1_Y \mathcal{X}_L) \\ & H^2(G, \mathrm{Div}_Y \mathcal{X}_L) \arrow[ur] & \end{tikzcd}$$

In the top half, that the horizontal map is an isomorphism comes from the case $i = 0$ of the diagram (2·5). Since the two other maps are injective, it follows that they are isomorphisms.

Therefore the composition $H^0(G, \mathbb{Z}) \to H^2(G, \mathrm{Div}_Y \mathcal{X}_L)$ is surjective, and hence so is the map (2·4). $\square$

2·3. *Completion of the proof*

We can now finish the proof of Theorem 1.

Proof of Theorem 1 Recall that the scheme $\mathcal{X}$ was constructed from the given model of

X by removing all of the special fibre apart from the open subset U. By construction, the geometric hypotheses of Proposition 6 are satisfied; and U has a $\mathbb{Z}_p$-point, for otherwise the theorem is vacuous. The conclusion is that there exists an algebra B in $\mathrm{Br}(K/\mathbb{Q}_p)$ such that $\phi_K(\mathcal{A} - B) = 0$.

Proposition 3 and its corollaries, applied to $\mathcal{A} - B$, complete the proof. □

3. *Examples*

3·1. *Construction of an Azumaya algebra on some diagonal quartic surfaces*

Let V be the diagonal quartic surface in $\mathbb{P}^3$ defined by the equation

$$a_0 X_0^4 + a_1 X_1^4 + a_2 X_2^4 + a_3 X_3^4 = 0 \tag{3·1}$$

where the a_i are rational integers. This is a K3 surface, hence has finitely generated Picard group; so it is possible to apply the methods described by Bright and Swinnerton-Dyer [1] to compute the algebraic part of the Brauer group.

When $a_0a_1a_2a_3$ is a square, Swinnerton-Dyer [17] describes a fibration of V over $\mathbb{P}^1$ in curves of genus 1 and uses this to produce Azumaya algebras on V by lifting elements of $\mathrm{Br}\, k(\mathbb{P}^1)$. When $a_0a_1a_2a_3$ is not a square, the same fibration itself defines an Azumaya algebra aplit by $\mathbb{Q}(\sqrt{a_0a_1a_2a_3})$, as described by Bright [2]. This may be summarised as follows:

PROPOSITION 8. *Let V be the surface* (3·1) *and suppose that $a_0a_1a_2a_3$ is not a square. Let θ denote a square root of $a_0a_1a_2a_3$. Then there exists a function f on V, defined over $\mathbb{Q}$, such that*

$$(f) = D + D' - 2\Pi$$

where D is a curve of genus 1 *defined over $\mathbb{Q}(\theta)$, D' is its conjugate, and Π is a plane section; and*

$$\mathcal{A} = (\mathbb{Q}(\theta)/\mathbb{Q}, f)$$

defines a non-trivial Azumaya algebra on V.

We may assume that f can be written as a rational function on $\mathbb{P}^3$, such that neither the numerator nor the denominator is divisible by any constant prime factor; this can clearly be achieved by multiplying f by a constant factor.

Proposition 6 is very simple in this case of a quadratic extension. If $\mathbb{Q}(\theta)/\mathbb{Q}$ is unramified at p, then it shows that our Azumaya algebra $\mathcal{A}$ can be made to satisfy the conditions of Proposition 3 by maybe multiplying the function f by p. If, on the other hand, the extension is ramified, then Proposition 6 is trivial: at any component $Y_{\mathbb{Q}(\theta)}$ of the special fibre, the function f has even valuation and so $\phi_{\mathbb{Q}(\theta)}(\mathcal{A}) = 0$.

We will now look at what happens at odd primes where V has bad reduction. We will assume that V has points everywhere locally, and that

$$0 \le v_p(a_i) \le 3 \text{ for all } i, p.$$

Furthermore, we may assume that no prime p divides more than two of the a_i. For we can always multiply all the a_i by p and then remove fourth powers; in this way we will always reach the stage where at most two of the a_i are divisible by p, *except* when the $v_p(a_i)$ are all different modulo 4. But in that case the surface is not locally soluble.

3·2. *Primes dividing one coefficient*

PROPOSITION 9. *Let p be an odd prime dividing precisely one of the coefficients a_i. Then*

(i) *the reduction of V at p is a cone over a smooth plane quartic curve C;*
(ii) *the reduction of f at p is the pullback of a function on C;*
(iii) *the vertex of the cone does not lift to a point of $V(\mathbb{Q}_p)$.*

Proof. Statement (i) is obvious. For statement (ii), consider the divisor of f, which is made up of curves of genus at most 1. The reduction of this modulo p will still be a union of curves of genus at most 1. By the Riemann-Hurwitz formula, the map from each of these components to the base curve C must be constant.

For statement (iii), suppose without loss of generality that p divides a_0. Then any lift of the vertex to $\mathbb{Q}_p$ is equivalent modulo p to $(1 : 0 : 0 : 0)$. So

$$v_p(a_1 X_1^4 + z_2 X_2^4 + a_3 X_3^4) \geq 4$$

but $v_p(a_0 X_0^4) < 4$, so the point does not lie on $V(\mathbb{Q}_p)$. □

In particular, the reduction of V at p is irreducible. Proposition 6 takes a very simple form: having removed any factors of p from the numerator and denominator of the rational function f, we have ensured that Proposition 3 applies.

COROLLARY 10. *If an odd prime p divides precisely one of the a_i, and does so to an even power, then* $\mathrm{inv}_p \mathcal{A}(x) = 0$ *for all $x \in V(\mathbb{Q}_p)$.*

Proof. In this case, $\mathbb{Q}(\theta)/\mathbb{Q}$ is unramified at p. Combine Proposition 9 with Corollary 4. □

COROLLARY 11. *If an odd prime p divides precisely one of the a_i, and does so to an odd power, then to evaluate $\mathcal{A}$ on $V(\mathbb{Q}_p)$ it suffices to evaluate f at the points of $C(\mathbb{F}_p)$.*

Proof. In this case, $\mathbb{Q}(\theta)/\mathbb{Q}$ is tamely ramified at p. Combine Proposition 9 with Corollary 5. □

Remark. In this case, the divisor cut out by f on the curve C is divisible by 2 in $\mathrm{Div}\, C$. Before evaluating f at each point of $C(\mathbb{F}_p)$, we should first check whether that divisor is in fact twice a principal divisor – if it is, then the reduction of f is a constant multiplied by a square and so $\mathrm{inv}_p \mathcal{A}$ is constant on $V(\mathbb{Q}_p)$.

As an example, consider the surface

$$9X_0^4 + 10X_1^4 = 12X_2^4 + 13X_3^4. \tag{3·2}$$

The construction described above gives the Azumaya algebra $\mathcal{A} = (\mathbb{Q}(\sqrt{390})/\mathbb{Q}, f)$ where

$$f = \frac{15X_0^2 + 12X_2^2 + 13X_3^2}{X_0^2}.$$

At $p = 5$, the equation reduces to that of a cone over a plane quartic curve. The extension is tamely ramified. To evaluate $\mathrm{inv}_5 \mathcal{A}$ at a point, we look to see whether the value of the function $(12X_2^2 + 13X_3^2)/X_0^2$ is square or non-square modulo 5. It turns out that both square and non-square values occur, and therefore there is no Brauer–Manin obstruction on the variety (3·2). The same happens at $p = 13$.

3·3. *Primes dividing two coefficients*

Let p be an odd prime dividing precisely two of the coefficients a_i. Then the reduction of V at p consists of four planes intersecting in a common line L. In this case, several different things can happen depending on the Galois structure of the four planes and whether f has zeros or poles on any of them.

On the surface (3·2) introduced above, the reduction at $p = 3$ is the union of four planes: two defined over $\mathbb{F}_3$, and the other two quadratic conjugates. The extension $\mathbb{Q}(\sqrt{390})/\mathbb{Q}$ is ramified at 3. The function defining the Azumaya algebra reduces to $(X_3/X_0)^2$, which is invertible and square on each of those planes, and hence $\mathrm{inv}_3 \mathcal{A}$ is 0 except possibly on the line $X_1 = X_3 = 0$ where the planes meet.

On the surface

$$X_0^4 + 13X_1^4 = 3X_2^4 + 13X_3^4, \tag{3·3}$$

the construction gives the Azumaya algebra $\mathcal{A} = (\mathbb{Q}(\sqrt{3})/\mathbb{Q}, f)$ where

$$f = \frac{-3X_0^2 - 26X_1^2 + 12X_2^2 + 13X_3^2}{X_0^2}.$$

At 13, the surface (3·3) reduces to four planes, each defined over $\mathbb{F}_{13}$. The function f is identically zero (with multiplicity 1) on two of those planes but not on the other two. Since the extension is unramified at 13, we deduce that $\mathrm{inv}_{13} \mathcal{A}$ is 0 at points of $V(\mathbb{Q}_{13})$ lying above one of the planes where f is not zero. At points lying above the other two planes, we deduce that $\mathrm{inv}_{13} \mathcal{A}$ is equal to $\mathrm{inv}_{13}(\mathbb{Q}(\sqrt{3})/\mathbb{Q}, 13) = 1/2$. We obtain no information about any points of $V(\mathbb{Q}_{13})$ lying above the line L.

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References

[**1**] M. Bright and Sir Peter Swinnerton-Dyer. Computing the Brauer–Manin obstructions. *Math. Proc. Cambridge Philos. Soc.*, 137(1):1–16, 2004. ISSN 0305-0041.

[**2**] M. J. Bright. Brauer groups of diagonal quartic surfaces. *J. Symbolic Computation*, 2005. To appear.

[**3**] J.-L. Colliot-Thélène and Sir Peter Swinnerton-Dyer. Hasse principle and weak approximation for pencils of Severi-Brauer and similar varieties. *J. Reine Angew. Math.*, 453:49–112, 1994. ISSN 0075-4102.

[**4**] J.-L. Colliot-Thélène, J.-J. Sansuc, and P. Swinnerton-Dyer. Intersections of two quadrics and Châtelet surfaces. I. *J. Reine Angew. Math.*, 373:37–107, 1987. ISSN 0075-4102.

[**5**] J.-L. Colliot-Thélène, J.-J. Sansuc, and P. Swinnerton-Dyer. Intersections of two quadrics and Châtelet surfaces. II. *J. Reine Angew. Math.*, 374:72–168, 1987. ISSN 0075-4102.

[**6**] J.-L. Colliot-Thélène, A. N. Skorobogatov, and Sir Peter Swinnerton-Dyer. Hasse principle for pencils of curves of genus one whose jacobians have rational 2-division points. *Inventiones Mathematicae*, 134:579–650, 1998.

[**7**] E. V. Flynn. The Hasse principle and the Brauer-Manin obstruction for curves. *Manuscripta Math.*, 115(4):437–466, 2004. ISSN 0025-2611.

[**8**] A. Grothendieck. Le groupe de Brauer I, II, III. In J. Giraud et al., editors, *Dix Exposés sur la Cohomologie des Schémas*, volume 3 of *Advanced studies in mathematics*, pages 46–188. North-Holland, Amsterdam, 1968.

[**9**] A. Grothendieck. *Séminaire de Géometrie Algébrique du Bois-Marie SGA 1: Revêtements Étales et Groupe Fondamental*. Number 224 in Lecture Notes in Mathematics. Springer-Verlag, 1971.

[**10**] D. Harari. Méthode des fibrations et obstruction de Manin. *Duke Math. J.*, 75(1): 221–260, 1994. ISSN 0012-7094.

[**11**] R. Hartshorne. *Algebraic Geometry*. Number 52 in Graduate Texts in Mathematics. Springer-Verlag, 1977.

[**12**] Yu. I. Manin. *Cubic forms: algebra, geometry, arithmetic*. North-Holland Publishing Co., Amsterdam, 1974. ISBN 0-7204-2456-9. Translated from Russian by M. Hazewinkel, North-Holland Mathematical Library, Vol. 4.

[**13**] Yu. I. Manin. Le groupe de Brauer–Grothendieck en géométrie diophantienne. In *Actes du Congrès International des Mathématiciens (Nice, 1970), Tome 1*, pages 401–411. Gauthier-Villars, Paris, 1971.

[**14**] J. S. Milne. Class field theory. Online notes, available on the author's web site at `http://www.jmilne.org/math`, 1997.

[**15**] J.-P. Serre. *Corps Locaux*, volume VIII of *Publications de l'Institut de Mathématique de l'Université de Nancago*. Hermann, Paris, 1968.

[**16**] Sir Peter Swinnerton-Dyer. Brauer-Manin obstructions on some Del Pezzo surfaces. *Math. Proc. Camb. Phil. Soc.*, 125:193–198, Jan. 1999.

[**17**] Sir Peter Swinnerton-Dyer. Arithmetic of diagonal quartic surfaces, II. *Proc. London Math. Soc.*, 80:513–544, 2000.