

Brauer–Manin obstructions on general diagonal quartic surfaces

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Thursday 28th October 2010



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Diagonal quartic surfaces

A **diagonal quartic surface** is a smooth surface $X \subset \mathbb{P}_{\mathbb{Q}}^3$ defined by an equation of the form

$$a_0X_0^4 + a_1X_1^4 + a_2X_2^4 + a_3X_3^4 = 0$$

where $a_0, a_1, a_2, a_3 \in \mathbb{Q}$.

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where $a_0, a_1, a_2, a_3 \in \mathbb{Q}$. We can assume that the a_i all lie in \mathbb{Z} and have no common factor, and that none of them is divisible by a fourth power. Diagonal quartic surfaces are **K3 surfaces**. Among K3 surfaces, diagonal quartics are convenient for numerical experiments because their geometry is straightforward: for example, there are 48 straight lines contained in \bar{X} which generate $\text{Pic } \bar{X} \cong \mathbb{Z}^{20}$.

Some statistics

There are **6714** diagonal quartic surfaces with a_0, a_1 positive and a_2, a_3 negative, and $1 \leq |a_0|, |a_1|, |a_2|, |a_3| \leq 15$.

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- Elsenhans, 2010: **130** have no rational point of height $\leq 10^7$.

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- There is also the possibility of an obstruction coming from the **transcendental** part of the Brauer group (Ieronymou).
- What can we say about the Brauer group and the Brauer–Manin obstruction “in general”?

A general diagonal quartic surface

Theorem

Let X be a diagonal quartic surface, and let H be the subgroup of $\mathbb{Q}^\times / (\mathbb{Q}^\times)^4$ generated by -1 , 4 and the quotients a_i/a_j .

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Let X be a diagonal quartic surface, and let H be the subgroup of $\mathbb{Q}^\times / (\mathbb{Q}^\times)^4$ generated by -1 , 4 and the quotients a_i/a_j . Suppose that:

- 1 $X(\mathbb{Q}_v) \neq \emptyset$ for all places v of \mathbb{Q} ;

Then $\text{Br } X / \text{Br } \mathbb{Q}$ has order 2, and there is no Brauer–Manin obstruction to the existence of rational points on X .

Outline of the proof

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- 2 Ensure that the Galois action on $\text{Pic } \bar{X}$ is “as general as possible” so that we know $H^1(\mathbb{Q}, \text{Pic } \bar{X})$ has order 2.
- 3 Write down explicitly a non-trivial Azumaya algebra \mathcal{A} on X , and find a prime p such that the evaluation map

$$X(\mathbb{Q}_p) \rightarrow \left(\frac{1}{2}\mathbb{Z}\right)/\mathbb{Z}, \quad Q \mapsto \text{inv}_p \mathcal{A}(Q)$$

is surjective, showing that \mathcal{A} gives no obstruction to the Hasse principle.

The transcendental Brauer group

To control the transcendental Brauer group, we use the following result. Recall that H is the subgroup of $\mathbb{Q}^\times/(\mathbb{Q}^\times)^4$ generated by -1 , 4 and the quotients a_i/a_j .

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Theorem (Ieronymou, Skorobogatov, Zarhin)

Let X and H be as before, and suppose that $H \cap \{2, 3, 5\} = \emptyset$. Then $\text{Br } X = \text{Br}_1 X$.

So we will simply add this hypothesis to our main theorem.

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- The 48 lines are all defined over the field

$$K = \mathbb{Q}(\epsilon, \sqrt[4]{a_1/a_0}, \sqrt[4]{a_2/a_0}, \sqrt[4]{a_3/a_0})$$

where ϵ is a primitive eighth root of unity. Kummer theory shows that $[K : \mathbb{Q}] = |H|$.

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- The “most general” case is when $[K : \mathbb{Q}]$ is maximal, i.e. there are no non-trivial relations between the fourth roots of the a_i/a_0 . In this case we calculate that $H^1(\mathbb{Q}, \text{Pic } \bar{X})$ has order 2.

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- 1 $X(\mathbb{Q}_v) \neq \emptyset$ for all places v of \mathbb{Q} ;
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- 3 $|H| = 256$;

Then $\text{Br } X/\text{Br } \mathbb{Q}$ has order 2, and there is no Brauer–Manin obstruction to the existence of rational points on X .

An Azumaya algebra

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Let $Y \subset \mathbb{P}_{\mathbb{Q}}^3$ be the smooth quadric surface defined by

$$a_0 Y_0^2 + a_1 Y_1^2 + a_2 Y_2^2 + a_3 Y_3^2 = 0.$$

There is a morphism $\phi: X \rightarrow Y$ given by $Y_i = X_i^2$. Since X is everywhere locally soluble, so is Y ; and, since Y is a quadric, it follows that Y has a rational point.

Lemma

Pick a point $P = [y_0, y_1, y_2, y_3] \in Y(\mathbb{Q})$, and let g be the linear form

$$g = a_0 y_0 Y_0 + a_1 y_1 Y_1 + a_2 y_2 Y_2 + a_3 y_3 Y_3$$

defining the tangent plane to Y at P .

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$$f = \phi^* g = a_0 y_0 X_0^2 + a_1 y_1 X_1^2 + a_2 y_2 X_2^2 + a_3 y_3 X_3^2$$

be the quadratic form obtained by pulling g back to X .

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be the quadratic form obtained by pulling g back to X . Write $\theta = a_0 a_1 a_2 a_3$. Then the quaternion algebra

$$\mathcal{A} = (\theta, f/X_0^2) \in \text{Br } k(X)$$

is an Azumaya algebra on X .

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is an Azumaya algebra on X . The class of \mathcal{A} in $\text{Br } X/\text{Br } \mathbb{Q}$ is independent of the choice of P .

Evaluating the obstruction

We are now seeking a prime p such that the evaluation map

$$X(\mathbb{Q}_p) \rightarrow \left(\frac{1}{2}\mathbb{Z}\right)/\mathbb{Z}, \quad Q \mapsto \text{inv}_p \mathcal{A}(Q)$$

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- The “least bad” reduction, and indeed the “most general” type of bad reduction, comes at odd primes dividing just one of the coefficients a_i .
- If such a prime divides the coefficient to an even power, then $\mathbb{Q}(\sqrt{\theta})/\mathbb{Q}$ will be unramified at p , and the evaluation map will again be constant.

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- 3 $|H| = 256$;
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Then $\text{Br } X / \text{Br } \mathbb{Q}$ has order 2, and there is no Brauer–Manin obstruction to the existence of rational points on X .

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as long as $f(Q) \in \mathbb{Z}_p^\times$, since $\mathbb{Q}(\theta)/\mathbb{Q}$ is tamely ramified at p .

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as long as $f(Q) \in \mathbb{Z}_p^\times$, since $\mathbb{Q}(\theta)/\mathbb{Q}$ is tamely ramified at p . This only depends on the reduction of $Q \bmod p$.

Reducing the problem mod p

Let \tilde{X} denote the reduction of X mod p , and \tilde{f} the reduction of the quadratic form f mod p . We have reduced the problem to the following:

Lemma

To show that there is no Brauer–Manin obstruction to the Hasse principle on X , it is enough to show that the quadratic form \tilde{f} takes both square and non-square values in \mathbb{F}_p^\times at points of $\tilde{X}(\mathbb{F}_p)$.

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Recall the definition of f :

$$f = a_0y_0X_0^2 + a_1y_1X_1^2 + a_2y_2X_2^2 + a_3y_3X_3^2.$$

Since $p \mid a_0$, the reduction \tilde{f} has no term in X_0^2 .

Reducing the problem mod p

The reduction \tilde{X} of X mod p also has no term in X_0 :

$$\tilde{X} : \tilde{a}_1 X_1^4 + \tilde{a}_2 X_2^4 + \tilde{a}_3 X_3^4 = 0.$$

Let $C \subset \mathbb{P}_{\mathbb{F}_p}^2$ be the plane curve defined by this equation.

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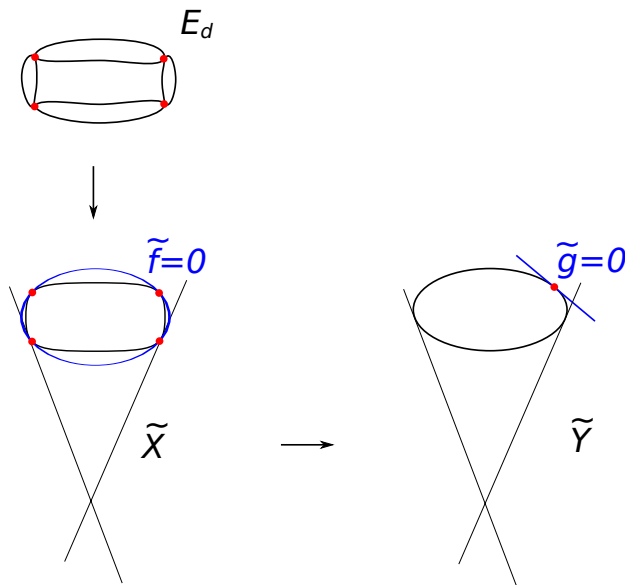
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Equivalently, we must show that each 2-covering $E_d \rightarrow C$ defined by

$$E_d : dZ^2 = \tilde{f},$$

for $d \in \mathbb{F}_p^\times / (\mathbb{F}_p^\times)^2$, has a smooth \mathbb{F}_p -point.

The geometric picture



The end of the proof

Each 2-covering $E_d \rightarrow C$ is singular above those points of C where \tilde{f} vanishes. However, it follows from the way that f was constructed that the normalisation of E_d is **unramified** over C , so has genus 5.

- If p is sufficiently large ($p > 114$ is certainly enough), then the Hasse–Weil bounds show that each E_d has a smooth \mathbb{F}_p -point, and we are finished.

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- Otherwise, there are only finitely many cases to check: for each p , there are finitely many curves C , and \tilde{f} only depends on C (and not the original X).
- The result turns out to be true **unless** $p \in \{7, 11, 17, 41\}$ **and** C is the Fermat curve (i.e. $\tilde{a}_1 = \tilde{a}_2 = \tilde{a}_3$).

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Theorem

Let X be a diagonal quartic surface, and let H be the subgroup of $\mathbb{Q}^\times / (\mathbb{Q}^\times)^4$ generated by -1 , 4 and the quotients a_i/a_j . Suppose that:

- 1 $X(\mathbb{Q}_v) \neq \emptyset$ for all places v of \mathbb{Q} ;
- 2 $H \cap \{2, 3, 5\} = \emptyset$;
- 3 $|H| = 256$;
- 4 there is an odd prime p which divides precisely one a_i , and does so to an odd power; *moreover, if $p \in \{7, 11, 17, 41\}$, then the reduction of X modulo p is not isomorphic to $x^4 + y^4 + z^4 = 0$.*

Then $\text{Br } X / \text{Br } \mathbb{Q}$ has order 2, and there is no Brauer–Manin obstruction to the existence of rational points on X .

An interesting consequence

In particular, we can construct whole families of diagonal quartic surfaces which have no Brauer–Manin obstruction to the Hasse principle.

Example

Let n be an integer, and suppose that there exists some prime $p > 11$ dividing n to an odd power. Then the diagonal quartic surface

$$2X_0^4 - 11X_1^4 - 7X_2^4 = nX_3^4$$

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- The curve $2x^4 - 11y^4 - 7z^4 = 0$ is everywhere locally soluble, so the surface has points everywhere locally.
- The condition on n ensures that the hypotheses on the group $H \subset \mathbb{Q}^\times / (\mathbb{Q}^\times)^4$ hold.

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The curve $2x^4 - 11y^4 - 7z^4 = 0$ actually has no rational points, so any rational point on our surface must have $X_3 \neq 0$.

An interesting consequence

The curve $2x^4 - 11y^4 - 7z^4 = 0$ actually has no rational points, so any rational point on our surface must have $X_3 \neq 0$. We deduce:

Theorem

Suppose that the Brauer–Manin obstruction is the only obstruction to the Hasse principle for K3 surfaces. Then every n as above can be represented as

$$n = 2x^4 - 11y^4 - 7z^4$$

for some $x, y, z \in \mathbb{Q}$.

Is this surprising?