Brauer–Manin obstructions on general diagonal quartic surfaces

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THE UNIVERSITY OF WARWICK

Outline

Diagonal quartic surfaces

A diagonal quartic surface is a smooth surface $X \subset \mathbb{P}^3_{\mathbb{Q}}$ defined by an equation of the form

$$a_0 X_0^4 + a_1 X_1^4 + a_2 X_2^4 + a_3 X_3^4 = 0$$

where $a_0, a_1, a_2, a_3 \in \mathbb{Q}$.

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where $a_0, a_1, a_2, a_3 \in \mathbb{Q}$. We can assume that the a_i all lie in $\mathbb{Z}$ and have no common factor, and that none of them is divisible by a fourth power. Diagonal quartic surfaces are K3 surfaces. Among K3 surfaces, diagonal quartics are convenient for numerical experiments because their geometry is straightforward: for example, there are 48 straight lines contained in $\bar{X}$ which generate $\operatorname{Pic} \bar{X} \cong \mathbb{Z}^{20}$.

Some statistics

There are 6714 diagonal quartic surfaces with a_0, a_1 positive and a_2, a_3 negative, and $1 \leq |a_0|, |a_1|, |a_2|, |a_3| \leq 15$.

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- Elsenhans, 2010: 130 have no rational point of height $\leq 10^7$.

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- There is also the possibility of an obstruction coming from the transcendental part of the Brauer group (Ieronymou).
- What can we say about the Brauer group and the Brauer–Manin obstruction "in general"?

A general diagonal quartic surface

Theorem

Let X be a diagonal quartic surface, and let H be the subgroup of $\mathbb{Q}^\times/(\mathbb{Q}^\times)^4$ generated by -1, 4 and the quotients a_i/a_j.

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Let X be a diagonal quartic surface, and let H be the subgroup of $\mathbb{Q}^\times/(\mathbb{Q}^\times)^4$ generated by -1, 4 and the quotients a_i/a_j. Suppose that:

1. *$X(\mathbb{Q}_v) \neq \emptyset$ for all places v of $\mathbb{Q}$;*

Then $\operatorname{Br} X / \operatorname{Br} \mathbb{Q}$ has order 2, and there is no Brauer–Manin obstruction to the existence of rational points on X.

Outline of the proof

At each step towards proving the desired result, we will need to impose conditions to ensure that X is "sufficiently general". The three steps are as follows.

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2. Ensure that the Galois action on $\operatorname{Pic} \bar{X}$ is "as general as possible" so that we know $\mathrm{H}^1(\mathbb{Q}, \operatorname{Pic} \bar{X})$ has order 2.
3. Write down explicitly a non-trivial Azumaya algebra $\mathcal{A}$ on X, and find a prime p such that the evaluation map

$$X(\mathbb{Q}_p) \to (\tfrac{1}{2}\mathbb{Z})/\mathbb{Z}, \qquad Q \mapsto \operatorname{inv}_p \mathcal{A}(Q)$$

is surjective, showing that $\mathcal{A}$ gives no obstruction to the Hasse principle.

To control the transcendental Brauer group, we use the following result. Recall that H is the subgroup of $\mathbb{Q}^\times/(\mathbb{Q}^\times)^4$ generated by -1, 4 and the quotients a_i/a_j.

The transcendental Brauer group

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Theorem (Ieronymou, Skorobogatov, Zarhin)

Let X and H be as before, and suppose that $H \cap \{2,3,5\} = \emptyset$. Then $\operatorname{Br} X = \operatorname{Br}_1 X$.

So we will simply add this hypothesis to our main theorem.

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The algebraic Brauer group

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- The geometric Picard group $\operatorname{Pic} \bar{X}$ is generated by the classes of the 48 straight lines on X, and so the structure of the algebraic Brauer group is determined entirely by the Galois action on the 48 lines.

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- The 48 lines are all defined over the field

$$K = \mathbb{Q}(\epsilon, \sqrt[4]{a_1/a_0}, \sqrt[4]{a_2/a_0}, \sqrt[4]{a_3/a_0})$$

  where ϵ is a primitive eighth root of unity. Kummer theory shows that $[K : \mathbb{Q}] = |H|$.

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- The "most general" case is when $[K : \mathbb{Q}]$ is maximal, i.e. there are no non-trivial relations between the fourth roots of the a_i/a_0. In this case we calculate that $\mathrm{H}^1(\mathbb{Q}, \operatorname{Pic}\bar{X})$ has order 2.

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Let X be a diagonal quartic surface, and let H be the subgroup of $\mathbb{Q}^\times/(\mathbb{Q}^\times)^4$ generated by -1, 4 and the quotients a_i/a_j. Suppose that:

1. $X(\mathbb{Q}_v) \neq \emptyset$ *for all places* v *of* $\mathbb{Q}$;
2. $H \cap \{2, 3, 5\} = \emptyset$;
3. $|H| = 256$;

Then $\operatorname{Br} X / \operatorname{Br} \mathbb{Q}$ *has order* 2, *and there is no Brauer–Manin obstruction to the existence of rational points on* X.

An Azumaya algebra

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Let $Y \subset \mathbb{P}^3_{\mathbb{Q}}$ be the smooth quadric surface defined by

$$a_0 Y_0^2 + a_1 Y_1^2 + a_2 Y_2^2 + a_3 Y_3^2 = 0.$$

There is a morphism $\phi \colon X \to Y$ given by $Y_i = X_i^2$. Since X is everywhere locally soluble, so is Y; and, since Y is a quadric, it follows that Y has a rational point.

An Azumaya algebra

Lemma

Pick a point $P = [y_0, y_1, y_2, y_3] \in Y(\mathbb{Q})$*, and let* g *be the linear form*

$$g = a_0 y_0 Y_0 + a_1 y_1 Y_1 + a_2 y_2 Y_2 + a_3 y_3 Y_3$$

defining the tangent plane to Y *at* P*.*

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$$f = \phi^* g = a_0 y_0 X_0^2 + a_1 y_1 X_1^2 + a_2 y_2 X_2^2 + a_3 y_3 X_3^2$$

be the quadratic form obtained by pulling g *back to* X.

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be the quadratic form obtained by pulling g *back to* X*. Write* $\theta = a_0 a_1 a_2 a_3$*. Then the quaternion algebra*

$$\mathcal{A} = (\theta, f/X_0^2) \in \operatorname{Br} k(X)$$

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is an Azumaya algebra on X*. The class of* $\mathcal{A}$ *in* $\operatorname{Br} X / \operatorname{Br} \mathbb{Q}$ *is independent of the choice of* P*.*

Evaluating the obstruction

We are now seeking a prime p such that the evaluation map

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- The "least bad" reduction, and indeed the "most general" type of bad reduction, comes at odd primes dividing just one of the coefficients a_i.
- If such a prime divides the coefficient to an even power, then $\mathbb{Q}(\sqrt{\theta})/\mathbb{Q}$ will be unramified at p, and the evaluation map will again be constant.

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as long as $f(Q) \in \mathbb{Z}_p^\times$, since $\mathbb{Q}(\theta)/\mathbb{Q}$ is tamely ramified at p. This only depends on the reduction of Q mod p.

Reducing the problem mod p

Let $\tilde{X}$ denote the reduction of X mod p, and $\tilde{f}$ the reduction of the quadratic form f mod p. We have reduced the problem to the following:

Lemma

To show that there is no Brauer–Manin obstruction to the Hasse principle on X, it is enough to show that the quadratic form $\tilde{f}$ takes both square and non-square values in $\mathbb{F}_p^\times$ at points of $\tilde{X}(\mathbb{F}_p)$.

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Recall the definition of f:

$$f = a_0 y_0 X_0^2 + a_1 y_1 X_1^2 + a_2 y_2 X_2^2 + a_3 y_3 X_3^2.$$

Since $p \mid a_0$, the reduction $\tilde{f}$ has no term in X_0^2.

Reducing the problem mod p

The reduction $\tilde{X}$ of X mod p also has no term in X_0:

$$\tilde{X} : \tilde{a}_1 X_1^4 + \tilde{a}_2 X_2^4 + \tilde{a}_3 X_3^4 = 0.$$

Let $C \subset \mathbb{P}^2_{\mathbb{F}_p}$ be the plane curve defined by this equation.

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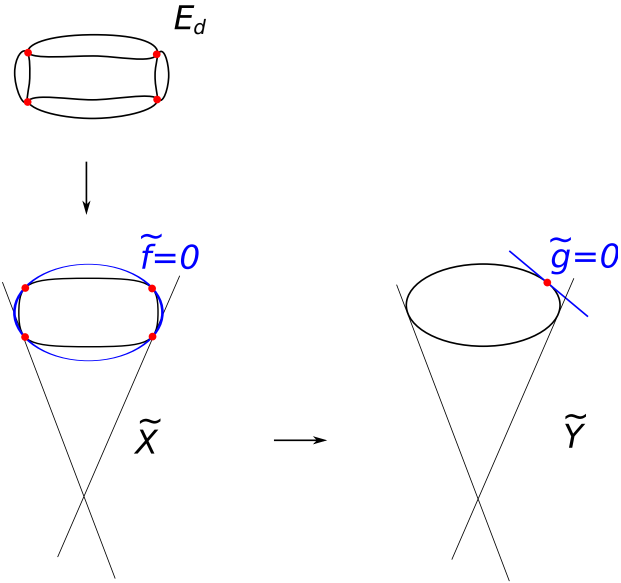
Equivalently, we must show that each 2-covering $E_d \to C$ defined by

$$E_d : dZ^2 = \tilde{f},$$

for $d \in \mathbb{F}_p^\times / (\mathbb{F}_p^\times)^2$, has a smooth $\mathbb{F}_p$-point.

The geometric picture



The end of the proof

Each 2-covering $E_d \to C$ is singular above those points of C where $\tilde{f}$ vanishes. However, it follows from the way that f was constructed that the normalisation of E_d is unramified over C, so has genus 5.

- If p is sufficiently large ($p > 114$ is certainly enough), then the Hasse–Weil bounds show that each E_d has a smooth $\mathbb{F}_p$-point, and we are finished.

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- Otherwise, there are only finitely many cases to check: for each p, there are finitely many curves C, and $\tilde{f}$ only depends on C (and not the original X).
- The result turns out to be true unless $p \in \{7, 11, 17, 41\}$ and C is the Fermat curve (i.e. $\tilde{a}_1 = \tilde{a}_2 = \tilde{a}_3$).

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Theorem

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1. $X(\mathbb{Q}_v) \neq \emptyset$ *for all places v of $\mathbb{Q}$;*
2. $H \cap \{2, 3, 5\} = \emptyset$;
3. $|H| = 256$;
4. *there is an odd prime p which divides precisely one a_i, and does so to an odd power; moreover, if $p \in \{7, 11, 17, 41\}$, then the reduction of X modulo p is not isomorphic to $x^4 + y^4 + z^4 = 0$.*

Then $\operatorname{Br} X / \operatorname{Br} \mathbb{Q}$ has order 2, and there is no Brauer–Manin obstruction to the existence of rational points on X.

An interesting consequence

In particular, we can construct whole families of diagonal quartic surfaces which have no Brauer–Manin obstruction to the Hasse principle.

Example

Let n be an integer, and suppose that there exists some prime $p > 11$ dividing n to an odd power. Then the diagonal quartic surface

$$2X_0^4 - 11X_1^4 - 7X_2^4 = nX_3^4$$

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- The curve $2x^4 - 11y^4 - 7z^4 = 0$ is everywhere locally soluble, so the surface has points everywhere locally.
- The condition on n ensures that the hypotheses on the group $H \subset \mathbb{Q}^\times/(\mathbb{Q}^\times)^4$ hold.

An interesting consequence

The curve $2x^4 - 11y^4 - 7z^4 = 0$ actually has no rational points, so any rational point on our surface must have $X_3 \neq 0$.

An interesting consequence

The curve $2x^4 - 11y^4 - 7z^4 = 0$ actually has no rational points, so any rational point on our surface must have $X_3 \neq 0$. We deduce:

Theorem

Suppose that the Brauer–Manin obstruction is the only obstruction to the Hasse principle for K3 surfaces. Then every n as above can can be represented as

$$n = 2x^4 - 11y^4 - 7z^4$$

for some $x, y, z \in \mathbb{Q}$.

Is this surprising?