Symmetry in Methods

Martin Bright

23 October 2002

1 Introduction

The aim of this article is to classify the various types of symmetry that a method might possess. We start by defining a method and a symmetry, and then proceed to list all possible types of symmetry which exist.

1.1 Definitions

A *row* on n bells is an element of the group of permutations of the set $\{1, \ldots, n\}$. A *change* on n bells is a permutation such that any bell moves at most one place: that is, a permutation c is a change if $|c(x) - x| \leq 1$ for all x from 1 to n. The *reverse* of a change c, which we will write $\bar{c}$, is defined to be $\tau c \tau$, where τ is the reversing permutation:

$$\tau(x) = n + 1 - x.$$

A *method* is a finite sequence $M = (c_1, \ldots, c_m)$ of changes. We require that the sequence not repeat: there is no non-zero integer k such that $c_{i+k} = c_i$ for all i. Two methods are said to be *equivalent* if they both consist of the same sequence of changes, but possibly 'rotated' to start from a different point in the sequence. That is, methods $M = (c_i)$ and $M' = (c'_i)$ are equivalent if they are of the same length m, and there exists an integer k such that $c'_i = c_{i+k}$ for all i, where indices are taken modulo m.

1.2 What is a symmetry?

What do we mean by a symmetry of a method? A method is usually thought of by some sort of representation in the plane, whether as a sequence of rows or as a geometrical diagram of lines. We will imagine the method drawn out vertically, with each row below the previous one. A line is drawn through the path of each bell, and the numbers representing the rows are removed. The lead is repeated over and over again, *ad infinitum*; so the resulting drawing forms a vertical strip infinite in both directions. Now we may apply some geometry. The notion of symmetry we use is that of an isometry of the plane which preserves the method we have drawn. Such an isometry must certainly preserve the vertical strip containing the method, and the group of these is generated by three types of elements: vertical reflections (that is, reflections in a horizontal line); horizontal reflections (in the vertical line which runs down the centre of the strip) and vertical translations.

With this in mind, we define the group G of method transformations which we will consider when studying symmetry. An element of G acts on a method to produce a new method. The group is generated by three elements:

- The vertical reflection V. If $M = (c_i)$, then $V(M) = (c'_i)$ where $c'_i = c_{m-i}$, m being the length of the method M.
- The horizontal reflection H. If $M = (c_i)$, then $H(M) = (\bar{c}_i)$.
- The translation T. Again, let $M = (c_i)$; then $T(M) = (c_{i+1})$.

Here, as always, indices are taken modulo the length m of the method.

The structure of G is determined by the following rules, where I represents the identity transformation:

- $H^2 = V^2 = I$;
- $HV = VH$;
- $HT = TH$;
- $VT = T^{-1}V$.

We can now define a *symmetry* of a method to be an element of G which maps the method to itself. The *symmetry group* of a method is the subgroup of G consisting of all symmetries of the method.

2 Classifying symmetries

Having defined the symmetry group of a method, we now go on to classify all those subgroups of G which can occur as the symmetry group of some method.

We do not want to consider every subgroup of G as giving rise to a different type of method symmetry group. For example, vertical reflection about the lead end should be considered the same as vertical reflection about any other change of the method: after all, these differ only by replacing the method with another equivalent one. The appropriate algebraic notion is that we wish to classify the subgroups of G up to conjugacy. In fact, it is easily seen that conjugating a subgroup by either H or V leaves it unchanged; the only non-trivial conjugations in G come from the translations, which arise precisely by replacing the method with an equivalent one.

There is one more constraint on the symmetry group of a method: it can never contain a non-trivial translation, that is, a non-trivial power of T. For, if it does, then the method violates our definition above: the method is in fact several leads of a shorter method.

2.1 Listing subgroups

The problem is now to list all the subgroups of G, not containing any non-trivial translations, up to conjugacy.

Denote by G' the subgroup of G generated by H and T. This subgroup is of index 2 in G, and therefore normal. It is also Abelian. Now let S be any subgroup of G. Then we denote by S' the intersection of S with G'. Either this is the whole of S, or S' is of index 2 in S. Therefore any subgroup S is generated either by a set of generators for S', or by a set of generators for S' together with some element of $G \setminus G'$.

To list the subgroups of G containing no non-trivial translation, we will perform the following steps:

1. List all subgroups S' of G' containing no non-trivial translation. Output each such S'.

2. For each such S', output $S = S'.\langle Vy \rangle$, where y runs through a set of representatives for G'/S' such that S contains no non-trivial translation.

For the first step, we note that, by the general theory of Abelian groups, S' needs at most two generators. Neither of these generators may be a translation T^i for i non-zero; for then S' would contain a translation. Therefore each generator is of the form HT^i for some non-zero i. But $(HT^i)^2 = T^{2i}$, so $2i$ must divide m, the order of T. If m is odd, this demands that i be zero; if m is even, we may also have $2i = m$. We cannot have both H and $HT^{m/2}$ in S', for then we also have $T^{m/2}$; so in fact S' can only have one generator. We deduce:

Proposition 1. *The possible subgroups S' are:*

$$\begin{array}{rl} \{I\}, \quad \{I,H\} & \textit{if } m \textit{ is odd;} \\ \{I\}, \quad \{I,H\}, \quad \{I, HT^{m/2}\} & \textit{if } m \textit{ is even.} \end{array}$$

For the second step, we must list the suitable elements y. We are only interested in S up to conjugacy. Conjugating by H and by V both leave subgroups unchanged; but conjugating by T replaces VT^i with VT^{i+2}. We are therefore only interested in elements $y = H^jT^i$ where i is considered modulo 2; so it is enough to take y to be one of I, H, T or HT. The last two possibilities are only necessary when m is even. This allows us to list all the possible symmetry groups.

Proposition 2. *The possible symmetry groups S of a method of length m are:*

$$\left.\begin{array}{r} \{I\} \\ \{I,V\} \\ \{I,VH\}, \\ \{I,H\} \\ \{I,H,V,VH\} \end{array}\right\} \quad \textit{if } m \textit{ is odd;}$$

$$\left.\begin{array}{r} \{I\} \\ \{I,V\} \\ \{I,VH\} \\ \{I,VT\} \\ \{I,VHT\}, \\ \{I,H\} \\ \{I,H,V,VH\} \\ \{I,H,VT,VHT\}, \\ \{I,HT^{m/2}\} \\ \{I,HT^{m/2},V,VHT^{m/2}\}, \\ \{I.HT^{m/2},VT,VHT^{1+m/2}\} \end{array}\right\} \quad \textit{if } m \textit{ is even.}$$

Proof. To form this list, we take each subgroup S' from Proposition 1. After listing $S = S'$ by itself, we try adjoining each element V, VH, VT and VHT to produce subgroups S. The case when m is odd is straightforward, as are those when m is even and the subgroup S' is either $\{I\}$ or $\{I, H\}$.

When we take $S' = \{I, HT^{m/2}\}$ and m even, there are four subgroups to try. These are

$$\begin{array}{ll} S_1 = \{I, HT^{m/2}, V, VHT^{m/2}\} & S_2 = \{I, HT^{m/2}, VT, VHT^{1+m/2}\} \\ S_3 = \{I, HT^{m/2}, VH, VT^{m/2}\} & S_4 = \{I, HT^{m/2}, VHT, VT^{1+m/2}\}. \end{array}$$

Now different things happen depending on whether m is congruent to 0 or 2 modulo 4. If m is congruent to 0 modulo 4, then S_1 is conjugate to S_3 and S_2 is conjugate to S_4. If, on the other hand, m is congruent to 2 modulo 4, then S_1 is conjugate to S_4 and S_2 to S_3. In either case, S_1 and S_2 between them represent both possible conjugacy classes. □

2.2 Listing the symmetries

Finally, we interpret these subgroups as symmetry types of methods. If the length of the method is odd, the five possible symmetry types are:

- no symmetry;
- vertical symmetry;
- horizontal symmetry;
- rotational symmetry, as Winter Major;
- both vertical and horizontal symmetry.

If the length of the method is even, there are eleven possible symmetry types:

- no symmetry;
- vertical symmetry about a change, as Plain Bob;
- rotational symmetry about a change, as Anglia Cyclic Bob Major;
- vertical symmetry about a row †;
- rotational symmetry about a row;
- horizontal symmetry;
- horizontal and vertical symmetry about a change, as Mirror Bob;
- horizontal and vertical symmetry about a row †;
- ‘glide reflection’ symmetry, as Double Eastern Bob;
- vertical symmetry about a change, and rotational symmetry about a point midway between the symmetry lines, as Bristol Surprise;
- vertical symmetry about a row, and rotational symmetry about a point midway between the symmetry lines †.

Note that several of these groups (those marked †) cannot possibly occur as the symmetry group of a true method: for any method whose changes are vertically symmetrical about a row must have two consecutive changes the same; thus the method is false. However, these groups may still occur when studying the symmetry of, say, the path of a single bell.